Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Monday 3 June 2024

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/02**

Further Mathematics

Advanced

PAPER 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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P 7 5 6 8 3 A 0 1 3 2

Pearson

1. (a) Using the definition of sinh x in terms of exponentials, prove that

$$4\sinh^3 x + 3\sinh x \equiv \sinh 3x$$

(2)

(b) Hence solve the equation

$$\sinh 3x = 19\sinh x$$

giving your answers as simplified natural logarithms where appropriate.

(5)

a) $4\sinh^3 x + 3\sinh x = 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ ①

$= 4\left(\frac{e^{3x} - 3e^{x} + 3e^{-x} + e^{-3x}}{8}\right) + \underbrace{3\left(\frac{e^x - e^{-x}}{2}\right)}_{3\left(\frac{4e^{x} - 4e^{-x}}{8}\right)}$

$= \frac{e^{3x} - e^{-3x}}{2} = \sinh 3x$ ① as required

b) $\sinh 3x = 19\sinh x$

$4\sinh^3 x + 3\sinh x = 19\sinh x$

$4\sinh^3 x - 16\sinh x = 0$

$4\sinh x(\sinh^2 x - 4) = 0$ ① $\sinh^{-1}\left(\frac{x}{a}\right) = \ln\{x + \sqrt{x^2 + a^2}\}$

$\sinh x = 0$
$x = 0$ ①

$\sinh^2 x = 4 \Rightarrow \sinh x = \pm 2$

$x = \ln(\pm 2 + \sqrt{(\pm 2)^2 + 1})$ ①

$x = \ln(2 + \sqrt{5})$ ① and $x = \ln(-2 + \sqrt{5})$ ①

P 7 5 6 8 3 A 0 2 3 2

Question 1 continued

(Total for Question 1 is 7 marks)

2. $$f(x) = \tanh^{-1}\left(\frac{3-x}{6+x}\right) \qquad |x| < \frac{3}{2}$$

(a) Show that

$$f'(x) = -\frac{1}{2x+3}$$

(4)

(b) Hence determine $f''(x)$

(1)

(c) Hence show that the Maclaurin series for $f(x)$, up to and including the term in x^2, is

$$\ln p + qx + rx^2$$

where p, q and r are constants to be determined.

(3)

a) let $u = \frac{3-x}{6+x}$ $\quad \frac{du}{dx} = \frac{-(6+x)-(3-x)}{(6+x)^2} = \frac{-9}{(6+x)^2}$ 1 1

$f(x) = \tanh^{-1} u$

Chain Rule with $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}$

$f'(x) = \frac{1}{1-u^2} \times \frac{du}{dx}$

$= \frac{1}{1-\left(\frac{3-x}{6+x}\right)^2} \times \frac{-9}{(6+x)^2}$ 1

$= \frac{\cancel{(6+x)^2}}{(6+x)^2-(3-x)^2} \times \frac{-9}{\cancel{(6+x)^2}}$

$= \frac{-9}{(6+x+3-x)(6+x-3+x)}$

$= \frac{-9}{9(2x+3)} = -\frac{1}{2x+3}$ 1 as required

b) $f'(x) = -(2x-3)^{-1}$, $f''(x) = 2(2x+3)^{-2}$ 1

$-1 \times -1 \times 2 \times (2x-3)^{-1-1}$

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Question 2 continued

c) $f(0) = \tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln 3$

$f'(0) = -\frac{1}{3}$

$f''(0) = \frac{2}{9}$ 1

up to x^2 term

$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0)..$ 1

$\therefore f(x) = \ln\sqrt{3} - \frac{1}{3}x + \frac{1}{9}x^2$ 1

(Total for Question 2 is 8 marks)

3. (a) Explain why

$$\int_{\frac{4}{3}}^{\infty} \frac{1}{9x^2+16}\,dx$$

is an improper integral.

(1)

(b) Show that

$$\int_{\frac{4}{3}}^{\infty} \frac{1}{9x^2+16}\,dx = k\pi$$

where k is a constant to be determined.

(4)

a) The upper limit is infinite ①

recognise that this is the form of $\frac{d}{dx}\tan^{-1}(x)$

b) $\displaystyle\int \frac{1}{9x^2+16}\,dx = \int \frac{1}{9\left(x^2+\frac{16}{9}\right)}\,dx = \frac{1}{9}\int \frac{1}{x^2+\left(\frac{4}{3}\right)^2}\,dx$ ①

$= \frac{1}{9} \times \frac{3}{4} \arctan\left(\frac{3x}{4}\right) = \frac{1}{12}\arctan\left(\frac{3x}{4}\right)$ ①

$\displaystyle\int_{4/3}^{\infty} \frac{1}{9x^2+16} = \lim_{t\to\infty}\left[\frac{1}{12}\arctan\left(\frac{3x}{4}\right)\right]_{4/3}^{t}$

$= \frac{1}{12}\left(\lim_{t\to\infty}\arctan\left(\frac{3t}{4}\right) - \arctan(1)\right)$ ①

$= \frac{1}{12}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{48}$ ①

as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}$

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Question 3 continued

(Total for Question 3 is 5 marks)

4. Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2}{(r+4)(r+6)} = \frac{n(an+b)}{30(n+5)(n+6)}$$

where a and b are integers to be determined.

(6)

$$\frac{2}{(r+4)(r+6)} = \frac{A}{r+4} + \frac{B}{r+6}$$

$$2 = A(r+6) + B(r+4)$$
$$2 = r(A+B) + (6A+4B)$$

comparing coefficients: $A+B=0$, $6A+4B=2$ } use $A=-B$ or $B=-A$
solve simultaneously: $A=1$, $B=-1$ ①

$$\therefore \frac{2}{(r+4)(r+6)} \equiv \frac{1}{r+4} - \frac{1}{r+6}$$ ①

$$\sum_{r=1}^{n} \frac{1}{r+4} - \frac{1}{r+6} :$$

when $r=1$: $1/5 - \cancel{1/7}$ will cancel out
$r=2$: $1/6 - \cancel{1/8}$
$r=3$: $\cancel{1/7} - \cancel{1/9}$
$r=4$: $\cancel{1/8} - \cancel{1/10}$
$\vdots$
$r=n-2$: $\cancel{1/n+2} - \cancel{1/n+4}$
$r=n-1$: $\cancel{1/n+3} - 1/n+5$
$r=n$: $\cancel{1/n+4} - 1/n+6$ ①

adding all together: $\frac{1}{5} + \frac{1}{6} - \frac{1}{n+5} - \frac{1}{n+6}$ ①

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Question 4 continued

$$\frac{1}{5}+\frac{1}{6}-\frac{1}{n+5}-\frac{1}{n+6}=\frac{11(n+5)(n+6)-30(n+6)-30(n+5)}{30(n+5)(n+6)}$$ 1

$$=\frac{11(n^2+11n+30)-30n-180-30n-150}{30(n+5)(n+6)}$$

$$=\frac{11n^2+61n}{30(n+5)(n+6)}=\frac{n(11n+61)}{30(n+5)(n+6)}$$ 1

Question 4 continued

Question 4 continued

(Total for Question 4 is 6 marks)

5. The locus C is given by

$$|z - 4| = 4$$

The locus D is given by

$$\arg z = \frac{\pi}{3}$$

(a) Sketch, on the same Argand diagram, the locus C and the locus D

(4)

The set of points A is defined by

$$A = \left\{z \in \mathbb{C} : |z - 4| \leqslant 4\right\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\}$$

(b) Show, by shading on your Argand diagram, the set of points A

(1)

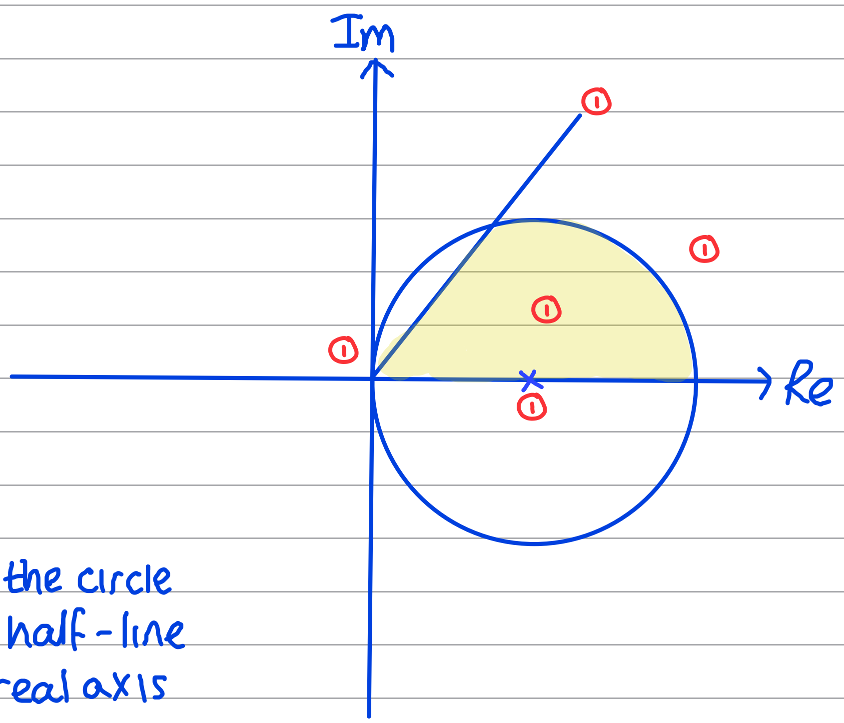
(c) Find the area of the region defined by A, giving your answer in the form $p\pi + q\sqrt{3}$ where p and q are constants to be determined.

(4)

a) $|z-4| = 4$: circle centre $(4,0)$, radius 4

$\arg z = \frac{\pi}{3}$: half-line from origin, angle $\frac{\pi}{3}$

b)



- within the circle
- below half-line
- above real axis

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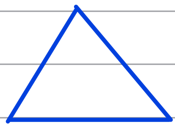
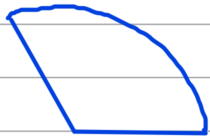
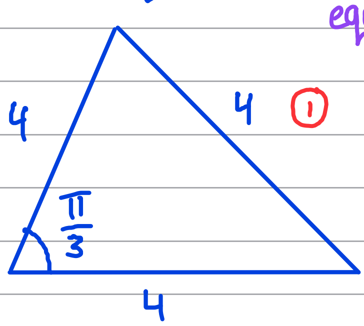
Question 5 continued

c) Area =  + 

Triangle: ← angle of $\frac{\pi}{3}$ means triangle must be equilateral.

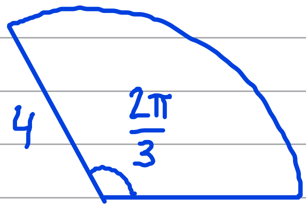


(1)

$A_1 = \frac{1}{2} \times 4 \times 4 \times \sin\frac{\pi}{3}$ (1)

$A_1 = 4\sqrt{3}$

Sector:



$A_2 = \frac{1}{2} \times \frac{2\pi}{3} \times 4^2 = \frac{16\pi}{3}$ (1)

$A_1 + A_2 = 4\sqrt{3} + \frac{16\pi}{3}$ (1)

P 7 5 6 8 3 A 0 1 3 3 2

Question 5 continued

Question 5 continued

(Total for Question 5 is 9 marks)

6. The motion of a particle P along the x-axis is modelled by the differential equation

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 4t + 12$$

where P is x metres from the origin O at time t seconds, $t \geqslant 0$

(a) Determine the general solution of the differential equation. **(6)**

(b) Hence determine the particular solution for which $x = 3$ and $\frac{dx}{dt} = -2$ when $t = 0$ **(3)**

(c) (i) Show that, according to the model, the minimum distance between O and P is $(2 + \ln 2)$ metres.

(ii) Justify that this distance is a minimum. **(4)**

For large values of t the particle is expected to move with constant speed.

(d) Comment on the suitability of the model in light of this information. **(1)**

a) auxillary equation: $2m^2 + 5m + 2 = 0 \leftarrow (2m+1)(m+2)$

$m = -0.5$ or $m = -2$ 1

complementary function: $x = Ae^{-0.5t} + Be^{-2t}$ 1

particular integral: let $x = pt + q$ 1

$\frac{dx}{dt} = p$, $\frac{d^2x}{dt^2} = 0$ substitute $\frac{dx^2}{d^2t}$ and $\frac{dx}{dt}$ in original equation

$2(0) + 5p + 2(pt + q) = 4t + 12$ solve for p and q

$2pt + (5p + 2q) = 4t + 12$ 1

compare coefficients: $2p = 4$ $5(2) + 2q = 12$

$p = 2$ $q = 1$ 1

general solution: $x = Ae^{-0.5t} + Be^{-2t} + 2t + 1$ 1

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Question 6 continued

b) $t=0, x=3$: $3 = Ae^0 + Be^0 + 2(0) + 1$

$\therefore A + B = 2$ ① (1)

$t=0$, $\frac{dx}{dt} = -2$:

$\frac{dx}{dt} = -0.5Ae^{-0.5t} - 2Be^{-2t} + 2$

$-2 = -0.5Ae^0 - 2Be^0 + 2$

$\therefore 0.5A + 2B = 4$ ②

solve ① and ② simultaneously: $A = 0$, $B = 2$ (1)

particular solution: $x = 2e^{-2t} + 2t + 1$ (1)

c) (i) find minimum by differentiating x and setting $=0$:

$\frac{dx}{dt} = -4e^{-2t} + 2 = 0$ (1)

$e^{-2t} = 0.5 = 2^{-1}$

$-2t = -\ln 2$

$t = \frac{1}{2}\ln 2$ (1)

sub $t = \frac{1}{2}\ln 2$ into x: $x = 2e^{-\ln 2} + \ln 2 + 1$
$= 2 + \ln 2$ (1)

(ii) $\frac{d^2x}{dt^2} = 8e^{-2t} > 0$ for all values of t, so the distance is a minimum (1)

Question 6 continued

d) for large values of t, $e^{-2t} \to 0$ so $x \to 2t+1$

$\therefore \frac{dx}{dt} = 2$ (= velocity)

so the speed becomes constant, and the model is suitable 1

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Question 6 continued

(Total for Question 6 is 14 marks)

7. (a) Determine the roots of the equation

$$z^6 = 1$$

giving your answers in the form $e^{i\theta}$ where $0 \leqslant \theta < 2\pi$

(2)

(b) Show the roots of the equation in part (a) on a single Argand diagram.

(2)

(c) Show that

$$\left(\sqrt{3} + i\right)^6 = -64$$

(2)

(d) Hence, or otherwise, solve the equation

$$z^6 + 64 = 0$$

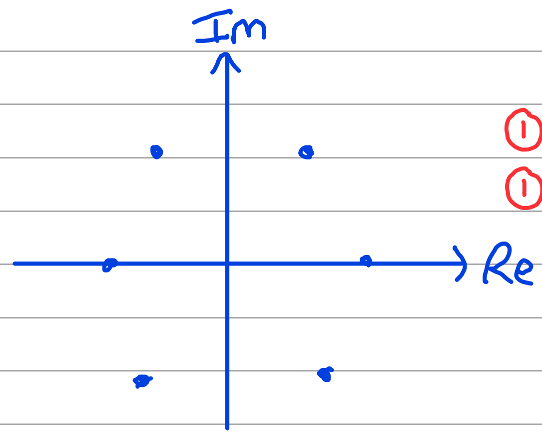
giving your answers in the form $re^{i\theta}$ where $0 \leqslant \theta < 2\pi$

(3)

a) $z^6 = 1 = e^{2k\pi i}$

$\therefore z = e^{\frac{k\pi i}{3}}$ 1 , $k = 0, 1, 2, 3, 4, 5$ 1

b)



1

1

c) $(\sqrt{3} + i)^6$ 1 let $z = \sqrt{3} + i$ $|z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore \left(2e^{\frac{\pi}{6}i}\right)^6 = 2^6 e^{\pi i} = -64$ 1

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Question 7 continued

d) know that $z=\sqrt{3}+i$ is a root

so let $z_1=\sqrt{3}+i=2e^{\frac{\pi i}{6}}$

to find other roots, multiply by $e^{\frac{k\pi i}{3}}$, $k=0,1,2,3,4,5$

$z_1=2e^{\frac{\pi i}{6}}$ ① for $r=2$

$z_2=2e^{\frac{\pi i}{6}}\times e^{\frac{\pi i}{3}}=2e^{\frac{\pi i}{2}}$

$z_3=2e^{\frac{\pi i}{6}}\times e^{\frac{2\pi i}{3}}=2e^{\frac{5\pi i}{6}}$

$z_4=2e^{\frac{\pi i}{6}}\times e^{\frac{3\pi i}{3}}=2e^{\frac{7\pi i}{6}}$ ① for at least 1 correct

$z_5=2e^{\frac{\pi i}{6}}\times e^{\frac{4\pi i}{3}}=2e^{\frac{3\pi i}{2}}$

$z_6=2e^{\frac{\pi i}{6}}\times e^{\frac{5\pi i}{6}}=2e^{\frac{11\pi i}{6}}$ ① for all roots correct

Question 7 continued

Question 7 continued

(Total for Question 7 is 9 marks)

8.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ k & 3 & 6 \end{pmatrix} \qquad k \neq 0$$

(a) Find, in terms of k, $\mathbf{A}^{-1}$

(4)

(b) Determine, in simplest form in terms of k, the coordinates of the point where the following planes intersect.

$$3x + y - z = 3$$
$$x + y + z = 1$$
$$kx + 3y + 6z = 6$$

(3)

a) $\det A = 3\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ k & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ k & 3 \end{vmatrix}$

$= 3(6-3) - (6-k) - (3-k)$
$= 2k$ 1

matrix of minors: $\begin{pmatrix} 3 & 6-k & 3-k \\ 9 & 18+k & 9-k \\ 2 & 4 & 2 \end{pmatrix}$

matrix of cofactors: $\begin{pmatrix} 3 & k-6 & 3-k \\ -9 & 18+k & k-9 \\ 2 & -4 & 2 \end{pmatrix}$ 1

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

transposed: $\begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}$

$A^{-1} = \dfrac{1}{2k}\begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}$ 1

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Question 8 continued

b) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2k}\begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$ ①

$$= \frac{1}{2k}\begin{pmatrix} 12 \\ 4k-24 \\ 12-2k \end{pmatrix} \begin{matrix} \leftarrow 3\times 3 + 1\times -9 + 2\times 6 \\ \leftarrow 3(k-6)+1(18+k)-4\times 6 \\ \leftarrow 3(3-k)+1(k-9)+2\times 6 \end{matrix}$$

$x = \frac{6}{k}$ ①, $y = \frac{2k-12}{k}$, $z = \frac{6-k}{k}$ ①

Turn over

Question 8 continued

Question 8 continued

(Total for Question 8 is 7 marks)

9.

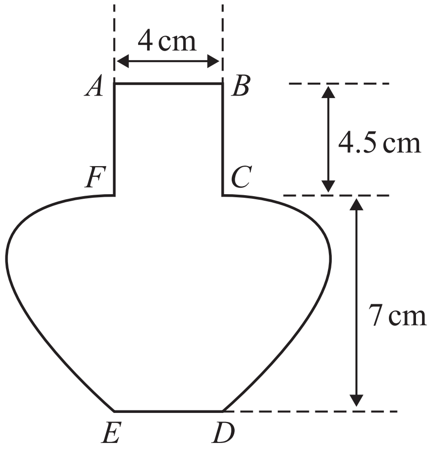

Figure 1

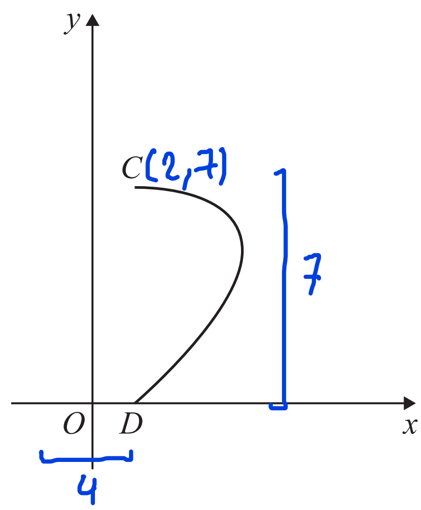

Figure 2

Figure 1 shows the central vertical cross-section *ABCDEFA* of a vase together with measurements that have been taken from the vase.

The horizontal cross-section between *AB* and *FC* is a circle with diameter 4 cm.

The base of the vase *ED* is horizontal and the point *E* is vertically below *F* and the point *D* is vertically below *C*.

Using these measurements, the curve *CD* is modelled by the parametric equations

$$x = a + 3\sin 2t \qquad y = b\cos t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

where a and b are constants and O is the fixed origin, as shown in Figure 2.

(a) Determine the value of a and the value of b according to the model.
(2)

(b) Using algebraic integration and showing all your working, determine, according to the model, the volume of the vase, giving your answer to the nearest cm^3
(7)

(c) State a limitation of the model.
(1)

a) from model, when $t = \frac{\pi}{2}$, $x = 2$ and $y = 7$

$x: 2 = a + 3\sin\pi$ $\quad$ $y: 7 = b\cos\pi$

$2 = a$ ① $\quad$ $7 = b$ ①

b) $V = \pi \int x^2 \frac{dy}{dt}\, dt$

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Question 9 continued

$x^2 = (2+3\sin 2t)^2 = 4 + 12\sin 2t + 9\sin^2 2t$

$\frac{dy}{dt} = -7\sin t$

$V = -7\pi \int (4 + 12\sin 2t + 9\sin^2 2t) \sin t \, dt$ ①

$= -7\pi \int (4\sin t + 12\sin 2t \sin t + 9\sin^2 2t \sin t) dt$

$= -7\pi \int (4\sin t + 24\sin^2 t\cos t + 36\sin^3 t\cos^2 t) dt$ ①

$= -7\pi \int (4\sin t + 24\sin^2 t\cos t + 36\sin t\cos^2 t(1-\cos^2 t)) dt$

$= -7\pi \int (4\sin t + 24\sin^2 t\cos t + 36\cos^2 t\sin t - 36\cos^4 t\sin t) dt$

$= -7\pi \left[-4\cos t + 8\sin^3 t - 12\cos^3 t + 7.2\cos^5 t \right]$ ① ①

adding limits: when $y=0$, $t=\frac{\pi}{2}$ when $y=7$, $t=0$

$V = -7\pi \left[-4\cos t + 8\sin^3 t - 12\cos^3 t + 7.2\cos^5 t \right]_{\frac{\pi}{2}}^{0}$ ①

$= \frac{588}{5}\pi$

Add volume of cylinder: $\pi \times 2^2 \times 4.5 = 18\pi$ ①

Total volume $= \frac{588}{5}\pi + 18\pi = 426 \text{ cm}^3$ (3sf) ①

Question 9 continued

c) The vase may not be completely smooth ①

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Question 9 continued

Question 9 continued

(Total for Question 9 is 10 marks)

TOTAL FOR PAPER IS 75 MARKS